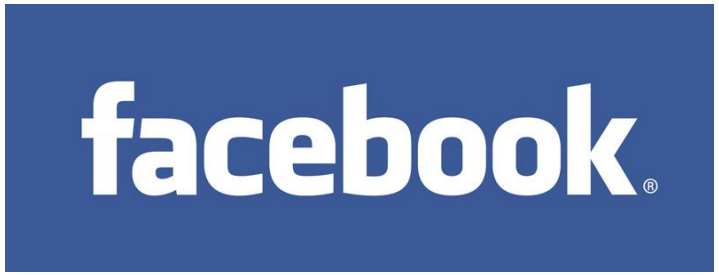
P vs. NP

Simpsons: Treehouse of Horror VI

Attribution

- These slides were prepared for the New Jersey Governor's School course "The Math Behind the Machine" taught in the summer of 2011 by Grant Schoenebeck
- Large parts of these slides were copied or modified from the previous years' courses given by Troy Lee in 2010 and Ryan and Virginia Williams in 2009.

Questions?



You are given the complete graph of Facebook.

What questions would you ask?

(What questions could we hope to answer?)

Polynomial Time

We have mentioned efficient algorithms several times in this course.

An efficient algorithm is generally taken to mean one whose running time depends polynomially on the size of the input.

Examples

Solving n linear equations in n unknowns.

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= b_1 \\
&\vdots \\
a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nn}x_n &= b_n
\end{aligned}
$$

Gaussian elimination---time about n^3

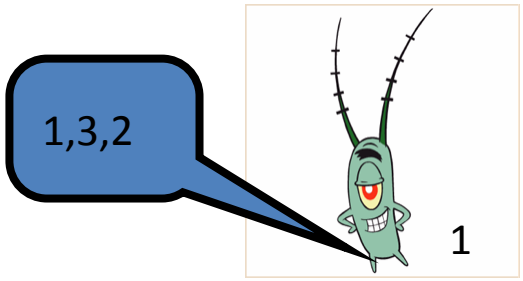
Examples

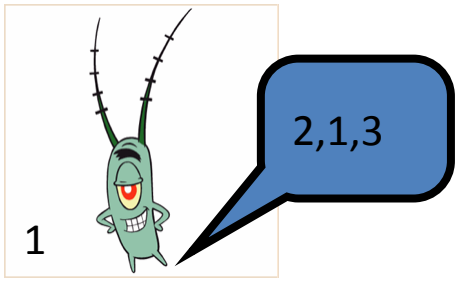
Linear programming problems.

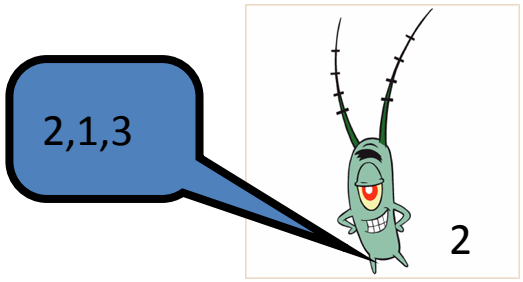
$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &\leq b_1 \\
a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &\leq b_2 \\
&\vdots \\
a_{n1}x_1 + a_{n2}x_2 + \ldots + \ldots a_{nn}x_n &\leq b_n \\
x_1, x_2, \ldots, x_n &\geq 0
\end{aligned}
$$

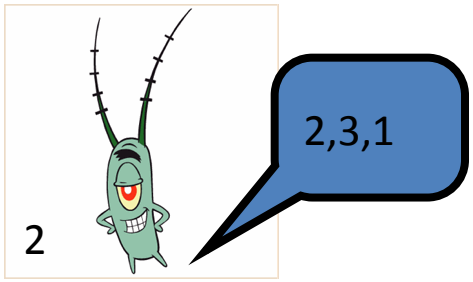
Time about n^4

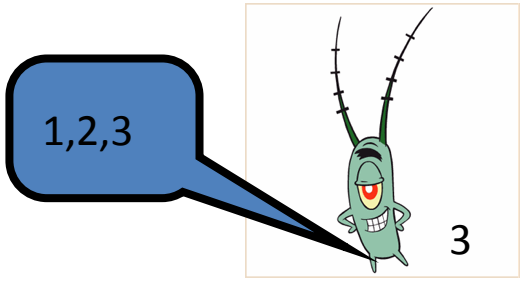
Examples

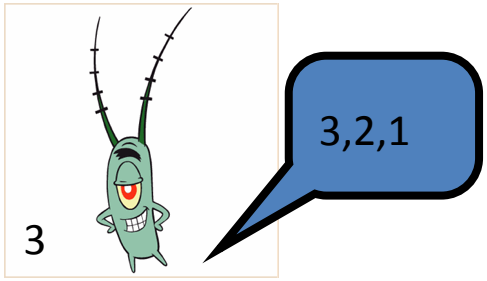












Find a stable pairing.

Time about n^3

Chinese Postman Problem

- Given a graph, what is the length of the shortest tour which traverses all edges?
  - Note that here you can---and may have to---use an edge more than once.

Reasonable Model?

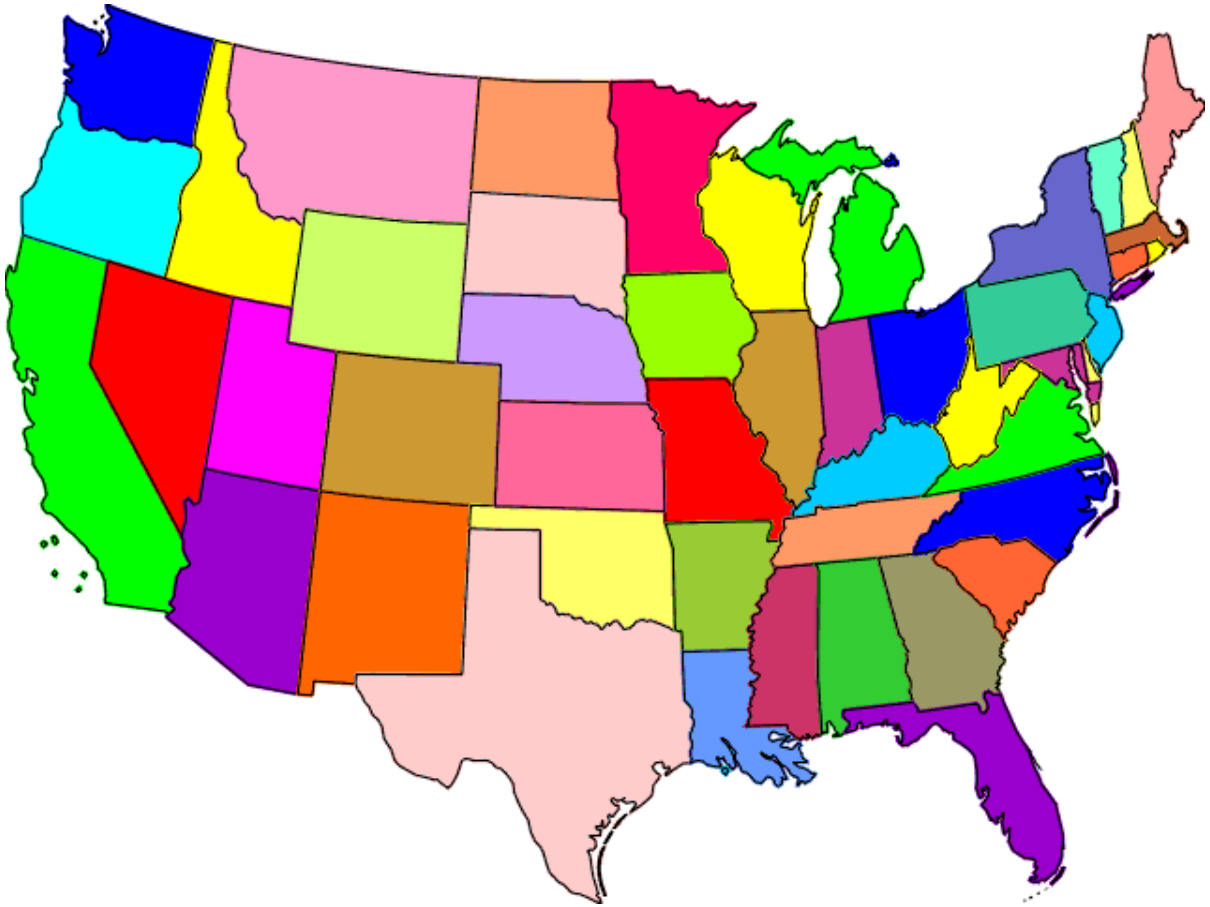
- An algorithm that takes $10^{67}n$ time would not be very practical...
- Even Gaussian elimination is currently not practical with a million unknowns.
- This model does offer very nice mathematical properties: for example, robustness and closure under composition.

What is NP?

Time for coloring...

Coloring Maps

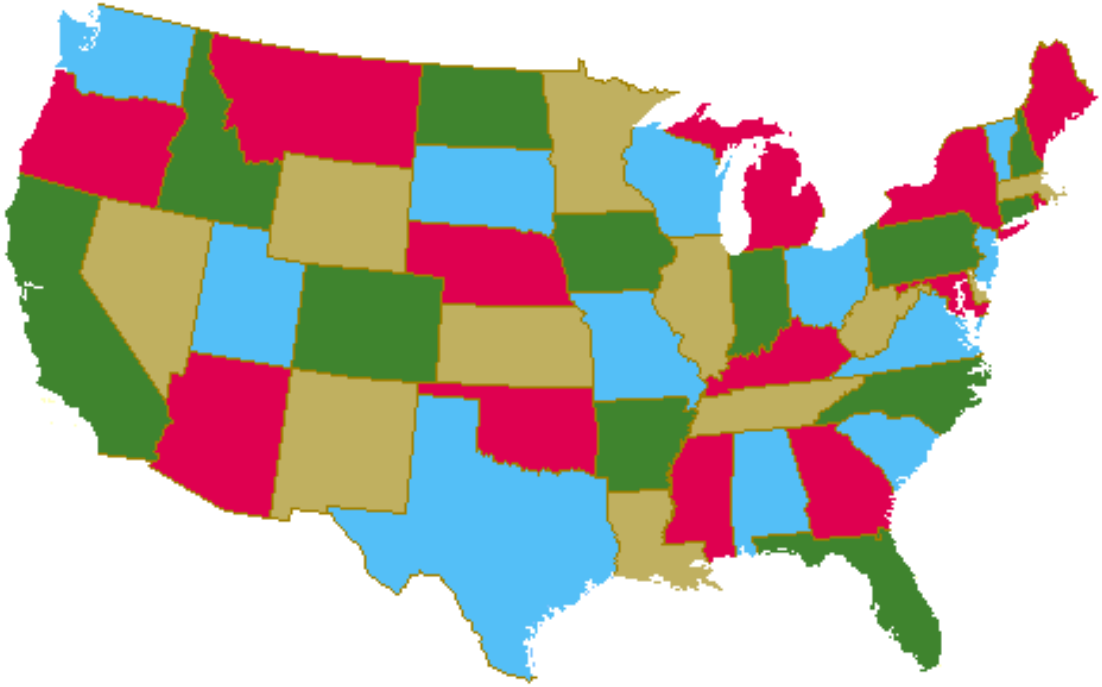
Try It!



In a map, don't want neighboring states to be the same color.
How many colors are needed?

Coloring Maps

How many colors did you need?



Are four colors necessary?

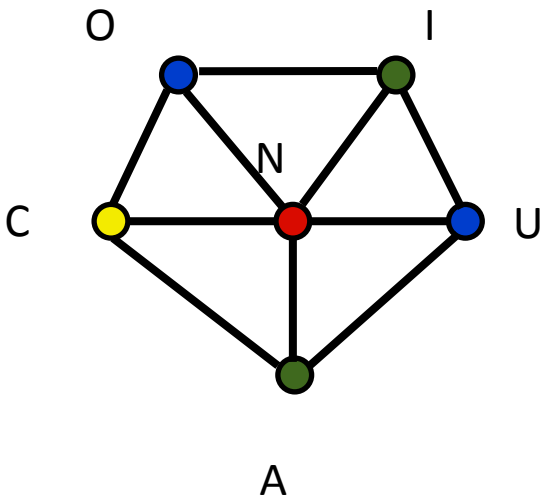
Look at the wild west...

Nevada has 5 nbhs.

The cycle of nbhs: oregon, idaho, utah, arizona, cali,oregon
requires 3 colors.

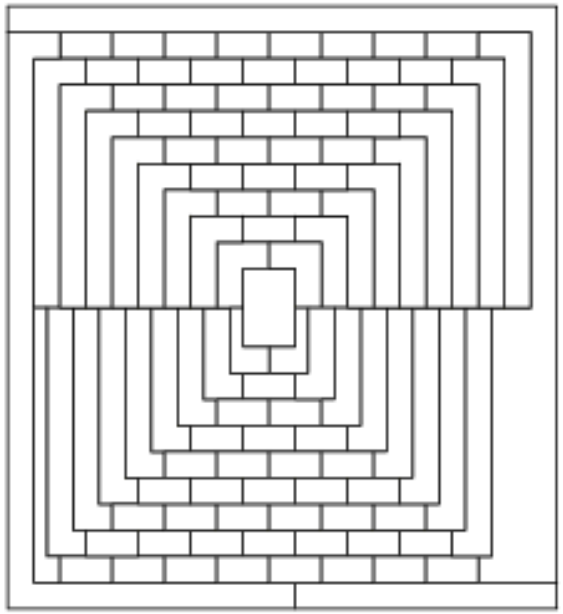
Are four colors necessary?

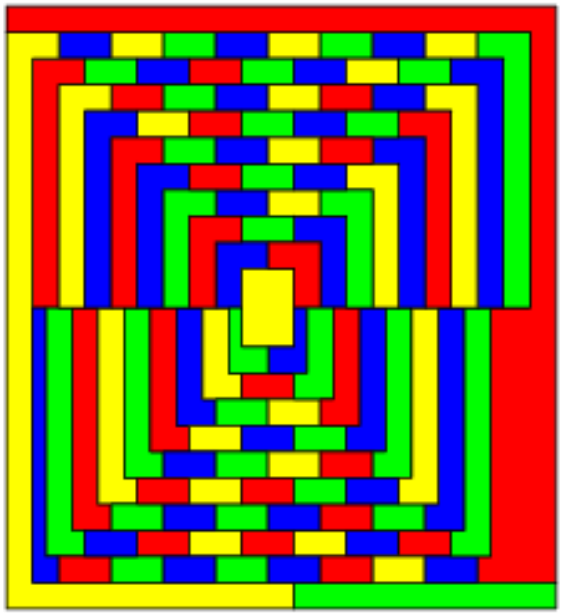
Look at the wild west...



Four Color Theorem

Every map can be colored with only four colors.





History of the four color theorem

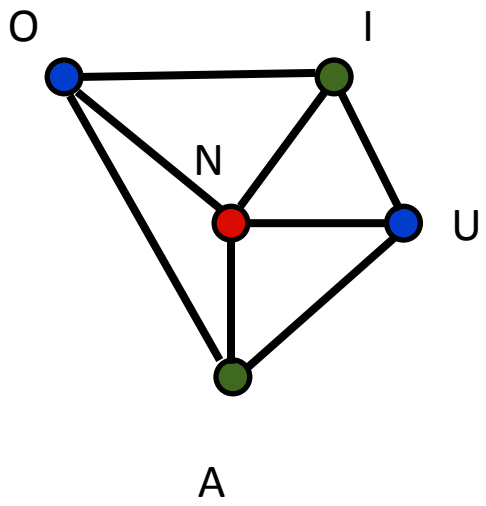
- 1852: Conjecture raised by Guthrie who noticed counties of England could be colored with only four colors.
- 1879: Proof given by Kempe.
- 1890: Heawood discovers flaw in Kempe's proof. Proves that five colors suffice.
- 1976: Appel and Haken prove it. Proof relies heavily on computer verification. Reduce problem to examining 1,936 special maps.

History of the four color theorem

- 1996: Sanders, Seymour, and Thomas give a simplified proof only requiring to check (by computer) 633 special maps.
- Their proof gives a n^2 time algorithm for four coloring a map.
- Still an open question to find a completely hand checkable proof!

Where we stand

- Every map can be colored with four colors.
- Some can be colored with three.



- Some can be colored with two. Can you tell which?

3-coloring problem

- Given a map, output “yes” if it can be colored with three colors, “no” otherwise.
- What is one possible algorithm to do this?
  - One could simply try all possible 3^n colorings. If a coloring “works,” it is easy to tell. Why coloring?
- Actually useful if:
  - Want to avoid conflicts.
  - Clustering by dissimilarity.

The Sixties

- Many different routing and scheduling problems did not have efficient solutions.
- They all seemed difficult but different nuts to crack.

NP

- **NP** is the class of problems which have solutions that can be efficiently verified.
  - As usual, efficiently means polynomial in size of input.
- **NP** stands for nondeterministic polynomial time.
- 3-coloring is in NP. Given a proposed coloring, we can quickly check if it works.

P vs. **NP**

- **P**: problems which we can efficiently solve.
- **NP**: problems which, given a proposed solution, we can efficiently check if it works.
- Every problem in **P** is also in **NP**.
- It is conjectured that there are problems in **NP**, for example 3-coloring, that are not in **P**.

P vs. **NP**

- Figuring out the relationship between P and NP is one of, if not the, greatest open problem in mathematics today.

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- It is one of the ~~7~~ open problems which the Clay Mathematics institute is offering $1,000,000 for its resolution.

Importance of **P** vs. **NP**

- Given a conjecture, find a proof.
- Given data on some phenomenon, find a theory explaining it.
- Given constraints (cost, strength, energy), find a design (bridge, medicine, phone).
- Code breaking.
- In each of these cases, when we see a good solution, we should be able to recognize it!

NP-completeness

- Around 1971, Cook and Levin developed the idea of **NP**-completeness. Soon after Karp showed 24 **NP**-complete problems off all shapes and colors.
- These are universal **NP**-problems...if you can solve them efficiently, you can solve ANY problem in **NP** efficiently.
- L is **NP**-complete if:
  - L is in **NP**
  - ANY other problem in NP reduces to L.
- If you come up with an efficient algorithm to 3-color a map, then **P**=**NP**.

NP-completeness

- Today over 3000 NP-complete problems known across all the sciences.
- Google Scholar search of NP-complete and biology returns over 10,000 articles.

NP-completeness

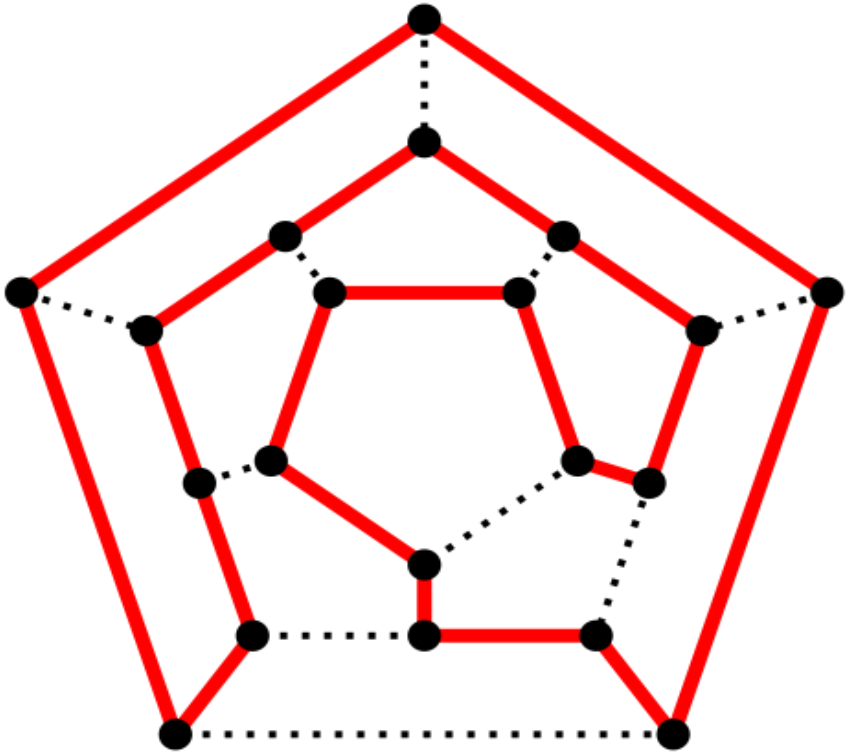
- Quintessential NP complete problem: 3 SAT.
- $(x_1 \vee \neg x_4 \vee x_5) \wedge (\neg x_1 \vee x_6 \vee \neg x_5) \wedge \cdots$
- Given a formula like this, does it have a Boolean assignment which makes it true?

Integer Programming

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &\leq b_1 \\
a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &\leq b_2 \\
&\vdots \\
a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nn}x_n &\leq b_n \\
x_1, x_2, \ldots, x_n &\in \{0, 1\}
\end{aligned}
$$

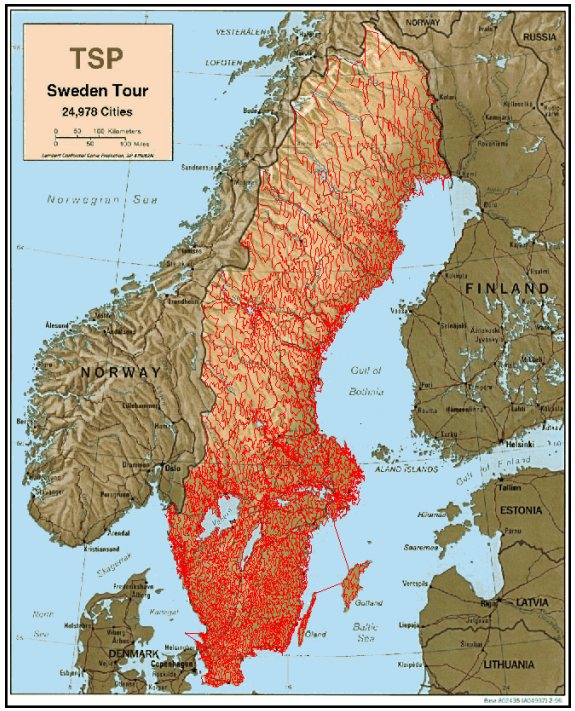
Hamiltonian Cycle

Given a graph, is there a cycle which visits all vertices exactly once?



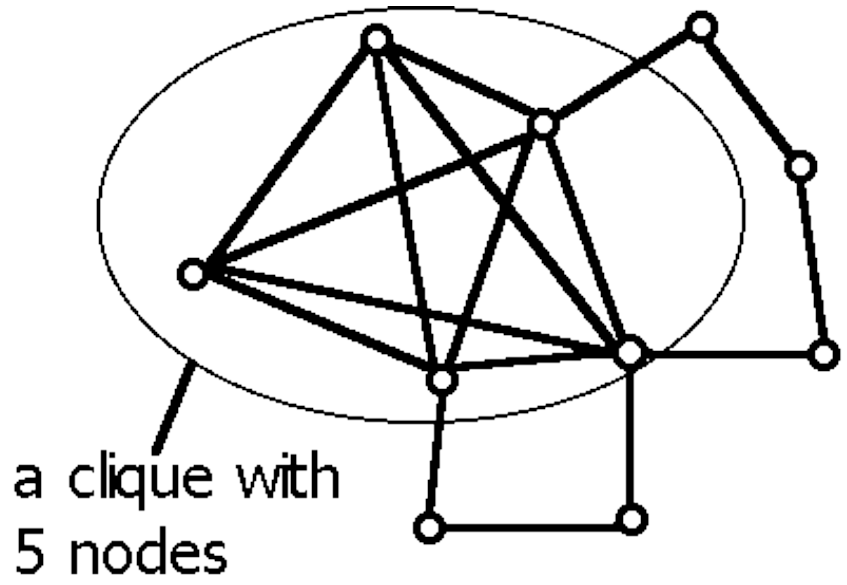
Traveling Salesman Problem

Given a list of cities and pairwise distances between them, is there a tour which visits each city exactly once and has length at most k?



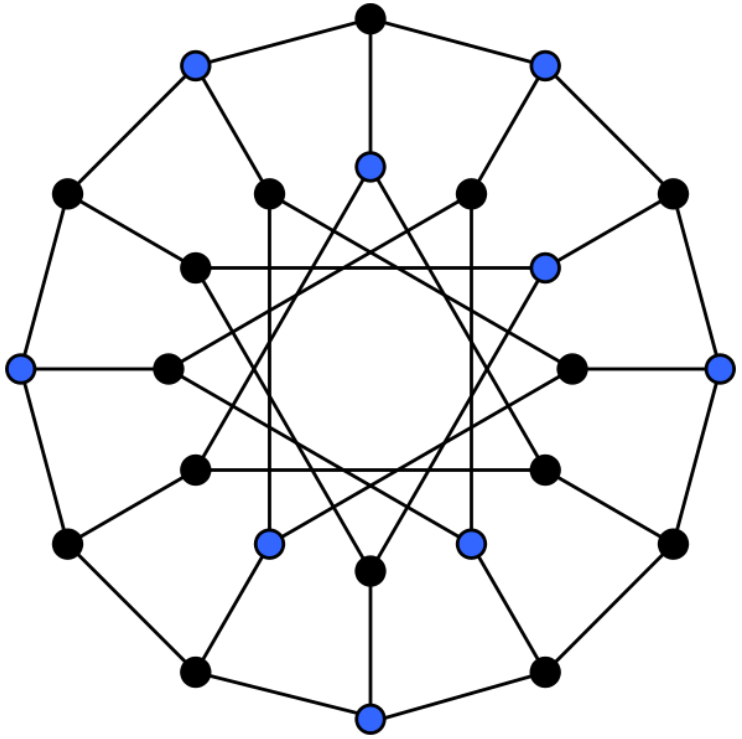
Clique

- Given (G, k) a graph and integer: Are there k nodes in G that are all connected to each other?



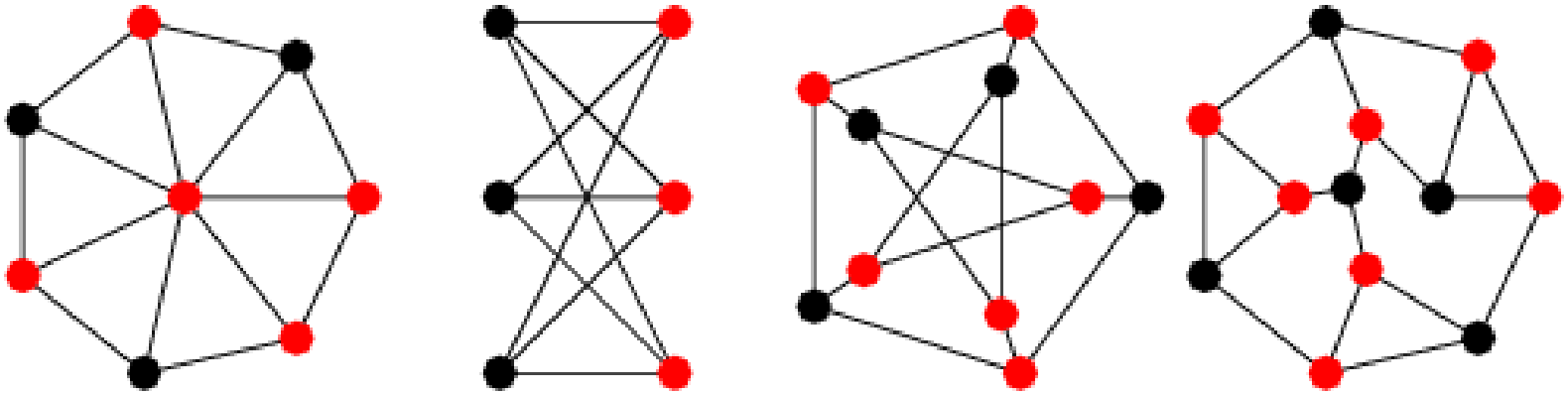
Independent Set

- Given (G, k) a graph and integer, are there k nodes in G none of which are connected to each other.



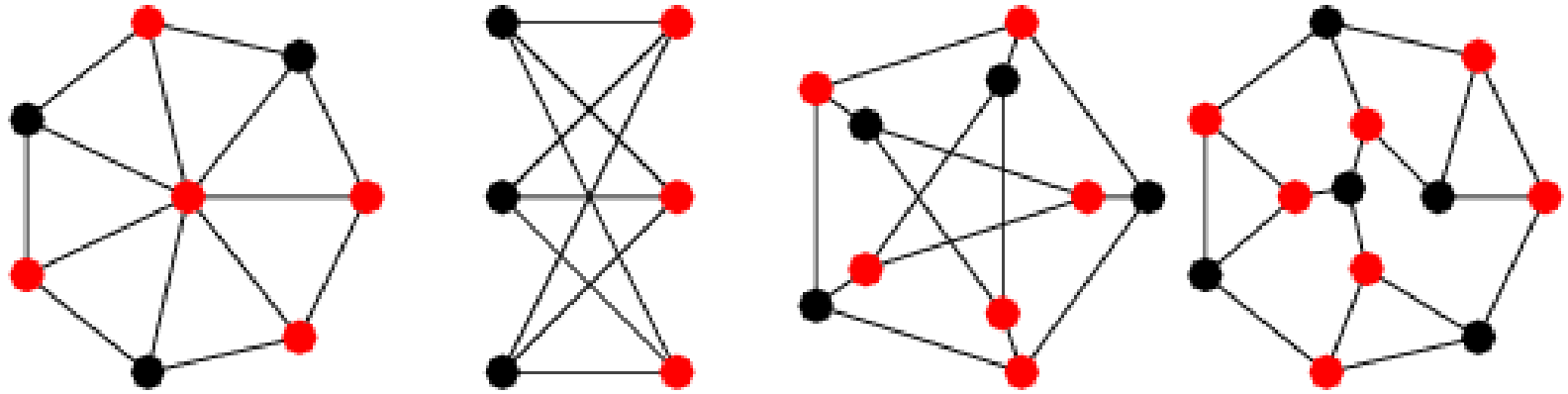
Vertex Cover

- Given (G, k) a graph and integer, are there k nodes in G none of which are connected to each other.



Set Cover

- Given list of items S and subsets of S: S_1, S_2, …, S_m and integer k: are there k subsets that "cover" S?
- Vertex Cover is a special case where each element is in exactly 2 sets.



NP reductions

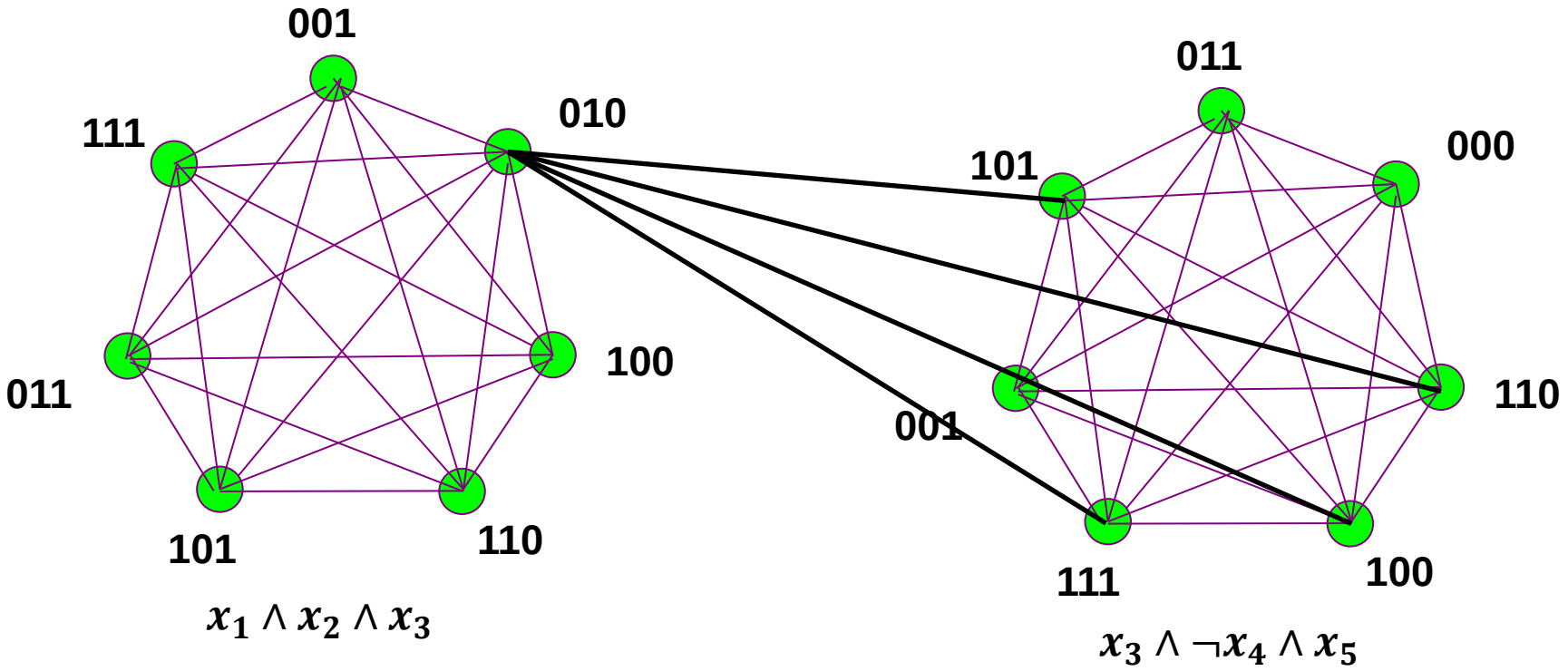
- 3-color ⇒ 3-SAT ⇒ Independent Set ⇒ Clique
- Independent Set ⇒ Vertex Cover ⇒ Set Cover
- Hamiltonian Cycle ⇒ Traveling Salesman Problem
- Idea: Given **NP**-complete problem A; problem B in **NP**. If A reduces to B than *every* **NP**-problem reduces to B. Therefore B is **NP**-complete.
- Implementation: Write an efficient procedure for A assuming a subroutine that solves B.

3-coloring to 3-SAT

- Given a 3-coloring problem G = (V, E), create 3-SAT problem with:
- Variables
  $x_{(v, c)}$ $\forall$ v $\in$ V, c $\in$ {g, r, b}
- Clauses
  - $\forall$(v,u) $\in$ E c $\in$ {g, r, b}
    $\neg x_{(v, c)} \vee \neg x_{(u, c)}$
  - $\forall$ v $\in$ V
    $\neg x_{(v, g)} \vee \neg x_{(u, r)} \vee \neg x_{(u, r)}$
    $x_{(v, g)} \vee \neg x_{(u, r)} \vee \neg x_{(u, r)}$
    $\neg x_{(v, g)} \vee x_{(u, r)} \vee \neg x_{(u, r)}$
    $\neg x_{(v, g)} \vee \neg x_{(u, r)} \vee x_{(u, r)}$
- 3-coloring: color vertices of graph g, r, b so that adjacent edges do not have same color
- 3-SAT: is there an assignment simultaneously satisfying all clauses

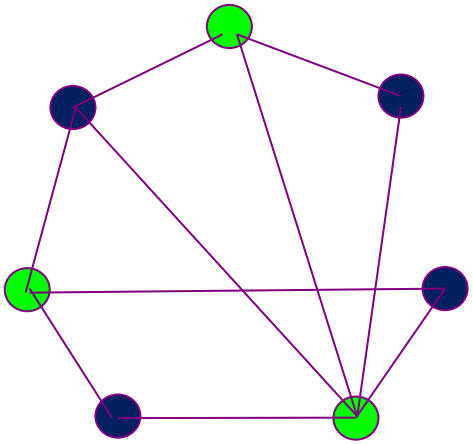
3-SAT to Independent Set

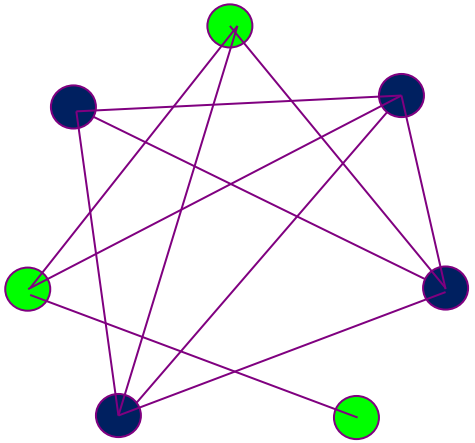
- Write procedure for 3-SAT given a subroutine computing Independent Set (G, k)
  - 3-SAT: is there an assignment simultaneously satisfying all clauses
  - Independent Set: Given (G, k) a graph and integer, are there k nodes in G none of which are connected to each other.



Independent Set to Clique

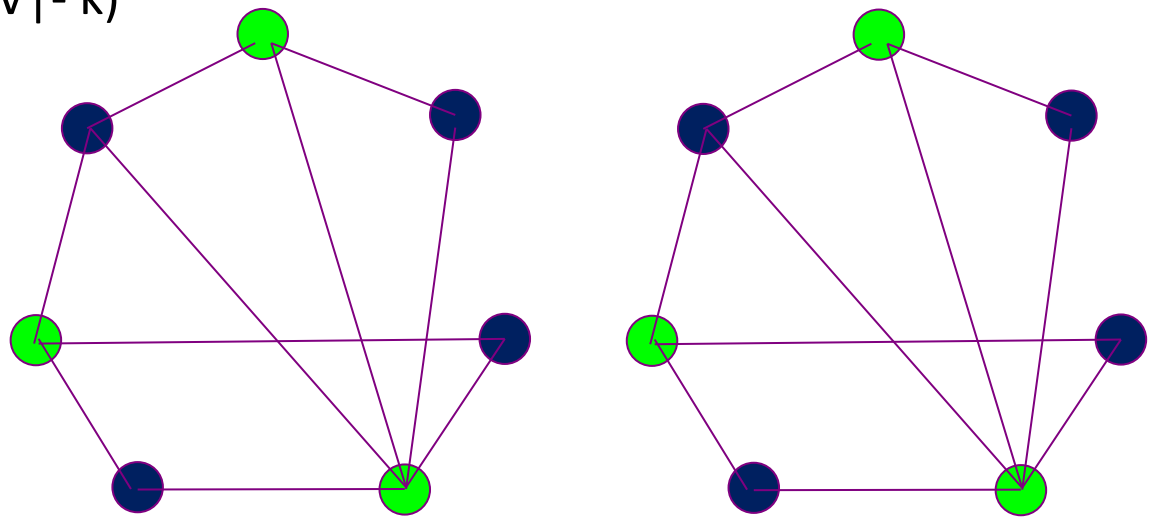
- Write an efficient procedure IndependentSet(G, k) given subroutine Clique(G', k'):
  - Independent Set: Given (G, k) a graph and integer, are there k nodes in G none of which are connected to each other.
  - Clique: Given (G, k) a graph and integer: Are there k nodes in G that are all connected to each other?
- Ask Clique (G', k) where G' is G with "opposite" edges.





Independent Set to Vertex Cover

- Write an efficient procedure IndependentSet(G, k) given subroutine VertexCover (G', k'):
  - Independent Set: Given (G, k) a graph and integer, are there k nodes in G none of which are connected to each other.
  - VertexCover: Given (G, k) a graph and integer: Are there k nodes in G that are incident on all edges?
- Given IndependentSet problem (G, k)
  Ask Vertex Cover (G, |V|- k)



Vertex Cover to Set Cover

- Write an efficient procedure VertexCover(G, k) given subroutine SetCover(S, S_1,…, S_m, k'):
  - VertexCover: Given (G, k) a graph and integer: Are there k nodes in G that are incident on all edges?
  - SetCover: Given list of items S and subsets of S: S_1, S_2, …, S_m and integer k: are there k subsets that "cover" S?

- Given VertexCover problem (G = (V, E), k), let S = E, $\forall v \in V \; let \;\; S_v = \{e \in E : e = (v, *)\}$, k' = k

Clique to ½-Clique

- Write an efficient procedure Clique(G, k) given subroutine ½-Clique(G).
  - ½ - Clique (G): Does G have |V|/2 nodes that all all connected to each other?
  - Clique(G, k): Given (G, k) a graph and integer: Are there k nodes in G that are all connected to each other?
- Given Clique problem (G, k) create ½-clique problem G' where you create |V| new vertices and attach |V|-k of them to every other vertex (new and old).

Hamiltonian Cycle to TSP

- Write a program the solves Hamiltonian Cycle
- You are have a subroutine TSP(all pair distances D, int k) that reports the solution to TSP
  - Hamiltonian Cycle: given a graph, is there a cycle which visits all vertices **exactly** once?
  - TSP: Given a list of cities and pairwise distances between them, is there a tour which visits each city exactly once and has length at most k?
- Given G = (V, E) create D where distance between u and v is
  - 1 if $(u,v) \in E$
  - 2 if $(u,v) \notin E$
- Report back TSP(D, |V|)